The slow motion of two or more spheres through a viscous fluid

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Expressions are derived for the velocity of two spheres, moving slowly under external forces through a viscous fluid, as a function of their separation and radii. They compare favourably with the available experimental data. A discussion of the interactions of three particles and some general comments on the settling of a swarm of spheres are also included.

1. Introduction

Despite the long time which has elapsed since Stokes's law was proved for the slow motion of a rigid sphere through a viscous fluid, there has been no connected account of the slow motion of two spheres. In this paper an attempt is made to remedy this omission for the very good reason that experiments are now being made (e.g. Hall 1956) of sufficient accuracy to justify a comparison with theory.

It should also be borne in mind that the results of these calculations are used as the basis for theories of sedimentation of collections of particles and of the flow of suspensions. A satisfactory account should, therefore, be of such a type that it can readily be developed to discuss the motion of many particles. The first calculation to be made (Smoluchowski 1911) was of this type and was used to estimate the motion of a cloud of particles. Smoluchowski calculated the first terms in an expansion in powers of the ratio of the particle radius to the distance between centres, and his results are valid only for small values of this parameter. Later Burgers (1942) extended this calculation to one higher order with the same purpose in mind, using arguments based to a large extent on physical intuition. In the meantime Stimson & Jeffery (1926) gave a complete solution of the problem of two spheres falling one behind the other, where there is axial symmetry, but their method uses the symmetry and cannot be extended to handle more than two particles.

The method described below is essentially an extension of that of Smoluchowski and Burgers. In the next three sections we discuss the solutions of the fluid equations that are needed for the solution of the two-body problem and comparison with experiment. It is not assumed that the spheres are equal, but §4 is mostly concerned with theoretical predictions for equal spheres, and the measurements which are needed to verify them. The method is easily extended to describe the motion of more than two particles and two later sections describe briefly some special results for three or more particles.

The essential features of the motion of two particles are predicted in the paper by Smoluchowski. Equal spheres falling under gravity fall together with a constant separation, but they only fall vertically when the line of centres is either horizontal or vertical; otherwise they tend to slide downwards along the line of the centres. The experimental results of Hall agree qualitatively with these calculations for large separations of the particles, the discrepancies being due almost certainly to experimental error. Hall also compares his results with the predictions of Stimson & Jeffery (1926). A complete theory, neglecting inertia terms, is obtained by combining the solution of Stimson & Jeffery with another, still to be obtained, of the problem where the line joining the spheres is perpendicular to the direction of motion.

A reader who is more interested in the qualitative results of this work need not examine too closely the next two sections which contain the mathematical analysis, although these sections define the notation used subsequently.

2. Solutions of the Navier–Stokes equations

The hydrodynamic equations for an incompressible viscous fluid, neglecting inertia terms, are

$$\nabla^2 u_\alpha = 2\partial p/\partial x_\alpha - (F_\alpha/\mu), \quad \sum_\alpha (\partial u_\alpha/\partial x_\alpha) = 0, \tag{2.1}$$

where $2\mu p$ is the mean pressure, μ is the coefficient of viscosity, $x_\alpha = (x, y, z)$ are Cartesian co-ordinates and u_α, F_α the Cartesian components of the velocity and body forces, respectively. If the body forces are zero, it is easily deduced that $\nabla^2 p = 0$, and the solutions can be written in the form $u_\alpha = v_\alpha + x_\alpha p$, where $\nabla^2 v_\alpha = 0$ (Kynch 1954). This result was used to derive the results stated below.

It is not difficult to find solutions of these equations such that two of the velocity components are zero and the third component varies in an arbitrary manner on the surface of the sphere $r = a$. If the non-zero component is $u_\beta = f$ at $r = a$, where f is a homogeneous function of the co-ordinates of degree m, which satisfies Laplace's equation, then the fluid motion is

$$u_\alpha = f\delta_{\alpha\beta} + \frac{r^2 - a^2}{2(1-m)} \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta}, \tag{2.2}$$

$$p = -\frac{2m-1}{2(m-1)} \frac{\partial f}{\partial x_\beta}. \tag{2.3}$$

For example, when $m = -(n+1)$, the function f can be any of the nth derivatives of $(1/r)$. If m is positive or zero the solutions are regular at the origin, and if m is negative they have a singularity at the origin.

To derive solutions corresponding to more general boundary conditions at $r = a$, we note that any function can be expanded on a sphere in terms of surface harmonics. In particular, if ϕ is any function defined on the surface, then

$$\phi = A\left(\frac{1}{r}\right) + A_\alpha \frac{\partial}{\partial x_\alpha}\left(\frac{1}{r}\right) + A_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{r}\right) + \dots \quad (\text{on } r = a), \tag{2.4}$$

where $A, A_\alpha, A_{\alpha\beta}, \dots$ are constants. In this equation the usual summation convention has been adopted, repeated suffices $\alpha, \beta, \dots$ being summed over their three values. To facilitate the manipulation of these expressions, a further convention is adopted where a group of n suffices $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ is denoted by a single roman letter (n), and a repeated suffix (n) is summed over all values of the group of n suffices and usually over all values of $n = 0, 1, 2, 3, \ldots$ as well. Thus equation (2.4) becomes

$$\phi = A(1/r) + A_\alpha (1/r)_\alpha + A_{\alpha\beta}(1/r)_{\alpha\beta} + \ldots$$
$$= \Sigma A_n (1/r)_n \quad \text{on} \quad (r = a),$$

i.e.
$$\phi = \Sigma A_n (1/a)_n. \tag{2.5}$$

Let the velocity components u_β have values, on a sphere A of radius a, expanded in the form,

$$u_\beta = \sum_m A_{\beta m}(1/a)_m \quad \text{on } A. \tag{2.6}$$

Let us also assume that the velocity tends to zero at infinity. Then by writing $f = (1/r)_m$ in the particular solutions (2.2), we derive the solution

$$u_\alpha = \sum_{\beta, m} A_{\beta m} S^\alpha_{\beta m}(A), \quad \text{where} \quad S^\alpha_{\beta m} = \left(\frac{1}{r}\right)_m \delta_{\alpha\beta} + \frac{r^2 - a^2}{2(m+2)} \left(\frac{1}{r}\right)_{\alpha\beta m},$$
$$p = -\sum_{\beta, m} A_{\beta m} \frac{2m+3}{2(m+2)} \left(\frac{1}{r}\right)_{\beta m}. \tag{2.7*}$$

The irrotational motion of a non-viscous fluid due to a moving body is usually described in terms of a suitable distribution of sources, dipoles and so on, whose velocity potentials are known. The solutions S can be regarded as those which arise from a generalization of this idea for viscous fluids. The radius a appears because we are specially interested in the motion of spherical particles, but it is simply an arbitrary parameter. The method used by Burgers (1942), which he attributes to Oseen (1927), seems to be based on these solutions when $a = 0$. The following interpretation of the pressure term is also due to Oseen. According to equation (2.1) any body force corresponds to a certain distribution of pressure and any term in the pressure can be regarded as the potential body force. Thus the motion (2.7) can be interpreted as the motion due to a distribution of body forces with a potential proportional to the pressure. If this motion is due to a moving body, the external force required to keep the body in motion through the viscous fluid is equivalent to a distribution of body forces throughout the fluid.

The expansion in terms of sources, dipoles and so on is not believed to be strongly convergent, and the use of a general notation which implies that a number of terms in the expansion are to be used, may, therefore, seem unnecessary. In fact, the solutions used here involving the radius of the sphere not only make it easier to satisfy the boundary conditions, but also seem to lead to expressions which converge quite rapidly unless $(r-a) \ll a$.

To illustrate the use of these results, let us consider two simple problems. The Stokes problem is that of a rigid sphere of radius a and velocity components V^A_α,

* In precisely the same way we can derive a solution with the same variation on the surface of the sphere but regular inside it, using the solutions $r^{2n+1}(1/r)_n$ of Laplace's equation. These solutions are homogeneous of degree n in the co-ordinates.

but no angular velocity, moving through a fluid otherwise at rest. If there is no slipping the solution is

$$u^0_\alpha = \sum_\beta a V^A_\beta S^\alpha_{\beta 0}(A). \tag{2.8}$$

To maintain the motion of the sphere it is necessary that there should be an external force with components,

$$F^A_\alpha = 6\pi\mu a V^A_\alpha, \tag{2.9}$$

It is often convenient to refer, not to the external force on a particle but the Stokes velocity, or the velocity derived from the force by this equation.

It can be shown, using the solutions (2.2) or in some other way, that in a general motion the force resisting the motion of a spherical particle is zero unless the fluid velocity contains the term $S^\alpha_{\beta 0}(A)$, i.e. a certain type of singularity in A, and that the external force necessary to balance the resistance is proportional to the coefficient of the term, in fact that

$$F_\alpha = 6\pi\mu A_{\alpha 0}. \tag{2.10}$$

The second problem is that of a sphere rotating with angular velocity $\boldsymbol{\omega}$. Since the surface velocity is $\boldsymbol{\omega} \wedge \mathbf{a}$, the boundary conditions are

$$\bar{A}_{xy} = \tfrac{1}{2}(A_{x,y} + A_{y,x}) = 0, \quad \hat{A}_{xy} = \tfrac{1}{2}(A_{x,y} - A_{y,x}) = -\omega_z a^3 \tag{2.11}$$

with similar equations for other pairs of suffixes. The solution is the elementary solution ($p = x, y, z$)

$$u_\alpha = \sum_{\beta, p} A_{\beta p}(1/r)_p \delta_{\alpha\beta} = (\boldsymbol{\omega} \wedge \mathbf{r})_\alpha / r^3. \tag{2.12}$$

A calculation of the forces on the particle shows that this motion is only maintained by an external couple with components

$$G_x = -8\pi\mu\hat{A}_{yz}, \text{ etc.} \tag{2.13}$$

Considerations similar to those for forces show that, in any general motion, the fluid only exerts a couple on the sphere when the constants $\hat{A}_{\alpha\beta}$ are not zero, and that the external couple applied to the sphere is given by the last equation.

In the next section of this paper it is necessary to expand in the neighbourhood of A the solutions $S^\alpha_{\beta m}(B)$, with a singularity at the centre B of a sphere of radius b.* These can be expressed in various ways using the formulae and the notation given in this paper and in Kynch (1956). If the position of the origin A relative to B is given by the vector $\mathbf{R} = \overrightarrow{BA}$, then

$$S^\alpha_{\beta,m}(B) = \sum_n \left\{\left(\frac{1}{R}\right)_{mn} \delta_{\alpha\beta} + \frac{d_n^2}{2(m+2)}\left(\frac{1}{R}\right)_{\beta m\alpha n} + \frac{1}{m+2} R_\epsilon \left(\frac{1}{R}\right)_{\beta m\alpha(n-\epsilon)}\right\} \frac{(-)^n}{n!!} r^{2n+1} \left(\frac{1}{r}\right)_n,$$

$$d_n^2 = R^2 - b^2 - (2m+3)r^2/(2n+3). \tag{2.14}$$

* It is in expansions of this type that it is better to use derivatives of (14) rather than associated Legendre polynomials. Such expansions have been used by Kirkwood in connexion with the transport and electrical properties of gases. The technique of using these expansions is not included here as it would involve explanations as long as those normally devoted to an account of the properties of expansions in terms of Legendre polynomials.

Derivatives of $(1/R)$ are calculated with respect to the co-ordinates of A, and we have used the notation

$$n!! = n!1.3.5\ldots(2n-1) = 2n!/2^n.$$

Values on the surface of a sphere of radius a about A are immediately obtained by putting $r = a$ in this equation. In the expression so obtained the term in $n = 1$, where we replace n by the single suffix p, deserves special mention. The coefficient of $(1/r)_p \equiv (1/a)_p$ has two suffices α and p, and we separate it into that part which does not change sign when these two are interchanged and that part which does change sign. These two are referred to as the symmetrical and anti-symmetrical parts of the coefficient, respectively, and we distinguish the latter by the use of square brackets and reserve round brackets for the symmetric part and the other coefficients. Thus, on $r = a$, we write

$$S^{\alpha}_{\beta m}(A) = (BA)^{\alpha 0}_{\beta m}(1/a) + \{(BA)^{\alpha p}_{\beta m} + [BA]^{\alpha p}_{\beta m}\}(1/a)_p + \sum_{n>1}(BA)^{\alpha n}_{\beta m}(1/a)_n. \quad (2.15)$$

3. The two-particle problem

The fluid motion due to two spheres A and B, when the distance R between their centres is large, is the sum of the motions due to each, i.e. a combination of two solutions already obtained in equation (2.10)

$$u_\alpha = A_{\beta 0} S^{\alpha}_{\beta 0}(A) + B_{\beta 0} S^{\alpha}_{\beta 0}(B). \quad (3.1)$$

The forces acting on the spheres are

$$F^A_\alpha = 6\pi\mu A_{\alpha 0}, \quad F^B_\alpha = 6\pi\mu B_{\alpha 0}. \quad (3.2)$$

In the limit of infinite separation, the velocities of the spheres are

$$U^A_\alpha = A_{\alpha 0}/a, \quad U^B_\alpha = B_{\alpha 0}/b. \quad (3.3)$$

Equation (3.1) is used as an approximation by both Smoluchowski (1911) and Burgers (1942). Smoluchowski assumes that the velocities of the particles are given and uses the equation to obtain an approximate expression for the applied forces. Using the expansion (2.19) for the second term of the equation (3.1) and equating the velocity of A to that term in the fluid velocity which does not vary over its surface, we find that, to this approximation,

$$aU^A_\alpha = A_{\alpha 0} + \sum_\beta B_{\beta 0}(BA)^{\alpha 0}_{\beta 0}. \quad (3.4a)$$

Similarly

$$bU^B_\alpha = B_{\alpha 0} + \sum_\beta A_{\beta 0}(AB)^{\alpha 0}_{\beta 0}. \quad (3.4b)$$

Smoluchowski solves these equations for the constants $A_{\alpha 0}$, $B_{\alpha 0}$, to a first approximation and derives the formulae

$$A_{\alpha 0} = aU^A_\alpha - bU^B_\beta(BA)^{\alpha 0}_{\beta 0}, \quad (3.5)$$
$$B_{\alpha 0} = bU^B_\alpha - aU^A_\beta(AB)^{\alpha 0}_{\beta 0}.$$

The forces are determined by equations (3.2).

Burgers (1942) assumes that the forces are known, and calculates the particle velocities. Thus $A_{\alpha 0}$, $B_{\alpha 0}$ are known and the velocities are given directly by equations (3.4). It is clear that this result is more accurate than the other. Inserting the values of the brackets and replacing forces by Stokes velocities (cf. 2.10), we deduce the formula,

$$U^A_\alpha = V^A_\alpha + (b/4R)\, V^B_\beta \{3(\delta_{\alpha\beta} + n_\alpha n_\beta) - (3n_\alpha n_\beta - \delta_{\alpha\beta})\,(a^2+b^2)/R^2\}. \tag{3.6}$$

$\mathbf{R} = \overrightarrow{BA}$, and n_α are the direction cosines of the directed line BA. Since the components of the Stokes velocity $\mathbf{V}^B$ along the line of centres are $V^B_R = V^B_\beta n_\beta$, the term $(V^B_\beta n_\beta)\, n_\alpha$ represents a velocity of magnitude V^B_R along that line. In vector form

$$\mathbf{U}^A = \mathbf{V}^A + \left(\frac{3b}{4R}\right)\mathbf{V}^B\left(1+\frac{a^2+b^2}{3R^2}\right) + \left(\frac{3b}{4R}\right)(\mathbf{V}^B.\mathbf{n})\,\mathbf{n}\left(1-\frac{a^2+b^2}{R^2}\right). \tag{3.7}$$

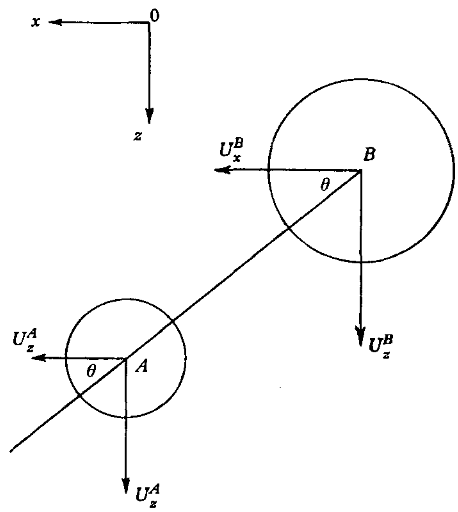

FIGURE 1. Co-ordinates and velocity components for two falling spheres.

When the particles fall under gravity, the last term is absent when the line of centres is horizontal: otherwise it represents a velocity *downwards* along the line of centres. The same is true for B, so that the particles not only fall with an increased velocity but move sideways in the same sense, as if it were easier to move along the line of centres than across it. This effect was first pointed out by Smoluchowski. If the line of centres is inclined at an angle to the horizontal (figure 1) the horizontal and vertical velocities are

$$\left.\begin{aligned} U^A_z &= V^A + \frac{3b}{4R}V^B\left[\left(1+\frac{a^2+b^2}{3R^2}\right) + \sin^2\theta\left(1-\frac{a^2+b^2}{R^2}\right)\right],\\ U^A_x &= \left(\frac{3b}{4R}\right)V^B \sin\theta\cos\theta\left(1-\frac{a^2+b^2}{R^2}\right). \end{aligned}\right\} \tag{3.8}$$

These expressions are not symmetric in the radii, so that the smaller of two unequal particles with the same Stokes velocities moves faster than the larger.

Burgers, by an argument based on equivalent body forces, deduces the first of further corrections which are needed to satisfy the boundary conditions. The nature of these corrections is found by expanding the approximation (3.1) on the surface of B,

$$u_\alpha^0 = B_{\alpha 0} + A_{\beta 0}\{(AB)^{\alpha 0}_{\beta 0}(1/b) + [AB]^{\alpha p}_{\beta 0}(1/b)_p + (AB)^{\alpha p}_{\beta 0}(1/b)_p + (AB)^{\alpha pq}_{\beta 0}(1/b)_{pq} + \dots\}. \tag{3.9}$$

As in the previous section, the symbols p and q are used to denote single indices, with three values. The fluid does not slip over B if B is given a translation corresponding to the first two terms, an angular velocity corresponding to the third term involving $[AB]^{\alpha p}_{\beta 0}$, and if in addition extra terms are added to the fluid velocity to cancel the remaining terms in the expansion. The most important of these is the fourth term, $A_{\beta 0}(AB)^{\alpha p}_{\beta 0}(1/b)_p$, which is cancelled on the surface of B by the correction

$$u_\alpha = -A_{\beta 0}(AB)^{\gamma p}_{\beta 0} S^\alpha_{\gamma p}(B). \tag{3.10}$$

Adding a similar term for the correction on the surface of A we obtain

$$u_\alpha^1 = A_{\beta 0} S^\alpha_{\beta 0}(A) - A_{\beta 0}(AB)^{\gamma p}_{\beta 0} S^\alpha_{\gamma p}(B) + B_{\beta 0} S^\alpha_{\beta 0}(B) - B_{\beta 0}(BA)^{\gamma p}_{\beta 0} S^\alpha_{\gamma p}(A). \tag{3.11}$$

Picking out terms which are constant on A, as before, we obtain for the velocity of A

$$aU_\alpha^A = A_{\alpha 0} + B_{\beta 0}(BA)^{\alpha 0}_{\beta 0} - A_{\beta 0}(AB)^{\gamma p}_{\beta 0}(BA)^{\alpha 0}_{\gamma p}. \tag{3.12}$$

The third term which is the correction term can be evaluated using the expressions

$$\left.\begin{aligned} (AB)^{\gamma p}_{\beta 0} &= (3a^3/4R^2)\, n_\beta(\delta_{\gamma p} - 3n_\gamma n_p) - (a^3/12)(3b^2+5a^2)(1/R)_{\beta\gamma p}, \\ \tfrac{1}{2}(BA)^{\alpha 0}_{\gamma p} + \tfrac{1}{2}(BA)^{\alpha 0}_{p\gamma} &= (a/2R^2)\, n_\alpha(\delta_{p\gamma} - 5n_p n_\gamma)(a/18)(3b^2+5a^2)(1/R)_{\alpha\gamma p}. \end{aligned}\right\} \tag{3.13}$$

Summing over γ and p, we find that the main part of the correction is

$$\Delta U_\alpha^A = -(15ab^3/R^4)(V_\beta^A n_\beta)\, n_\alpha. \tag{3.14}$$

This correction somewhat reduces the speed of A downwards along the line of centres given by equation (3.9). It is the correction given by Burgers (1942).

It is now clear that it is possible to use our equations to derive systematically all the necessary corrections to any order. The fluid velocity must have the form

$$u_\alpha = u_\alpha(A) + u_\alpha(B), \quad p = p(A) + p(B), \tag{3.15}$$

where $\quad u_\alpha(A) = \sum_{\gamma, m} A_{\gamma m} S^\alpha_{\gamma m}(A) \quad$ and $\quad u_\alpha(B) = \sum_{\gamma, m} B_{\gamma m} S^\alpha_{\gamma m}(B).$

According to equations (2.10) and (2.13) of the previous section, the constants $A_{\alpha 0}$, $B_{\alpha 0}$ are proportional to the external forces and the antisymmetrical parts $\hat{A}_{\alpha p}$, $\hat{B}_{\alpha p}$ $(p = 1)$ to the external couples on the particles A and B. The latter are zero in the absence of body couples and hence the coefficients $A_{\alpha p}$, $B_{\alpha p}$ are symmetrical in the two suffices.

The remaining coefficients are determined by the boundary conditions. For example, on the surface of A

$$u_\alpha = U_\alpha^A - \mathscr{A}_{\alpha\gamma p}\,\omega_\gamma^A a^3 (1/a)_p,$$

where U^A and ω^A are the translational velocity and angular velocity of A, and $\mathscr{A}_{\alpha\gamma p}$ is the alternating tensor. Expanding the expressions in equation (3.15) on the surface of A with the end of the previous section we obtain equations which are satisfied if*

$$A_{\alpha 0}+\Sigma B_{\gamma m}(BA)^{\alpha 0}_{\gamma m} = aU^A_\alpha, \tag{3.16}$$

$$\hat{A}_{\alpha p}+\Sigma B_{\gamma m}[BA]^{\alpha p}_{\gamma m} = \mathscr{A}_{\alpha\gamma p}\,\omega^A_\gamma a^3, \tag{3.17}$$

$$\bar{A}_{\alpha p}+\Sigma B_{\gamma m}(BA)^{\alpha p}_{\gamma m} = 0, \tag{3.18}$$

$$A_{\alpha n}+\Sigma B_{\gamma m}(BA)^{\alpha n}_{\gamma m} = 0 \quad (n > 1). \tag{3.19}$$

The bracket notation is explained at the end of § 2.

When the external forces are known, the unknown constants $A_{\alpha n}$ are obtained by solving equations (3.18) and (3.19) and the corresponding set of equations for the particle B. The velocity and angular velocity of the particles are then derived from equations (3.16) and (3.17).

These equations were solved by successive approximations, no more satisfactory method having been found, for the velocities of the spheres in terms of the forces acting upon them.† Since the expressions $(BA)^{\alpha n}_{\gamma m}$ are proportional to various powers of the ratio of the particle radii to the distances between centres the expansion is approximately but not strictly in powers of this ratio. No attempt was made to collect all the terms of a given power, as it was quite clear that better convergence is obtained by retaining the grouping corresponding to the brackets (BA).

The translation velocity of A is such that

$$aU^A_\alpha = A_{\alpha 0}+B_{\beta 0}(BA)^{\alpha 0}_{\beta 0}-A_{\beta 0}(AB)^{\gamma m}_{\beta 0}(BA)^{\alpha 0}_{\gamma m}+\dots \quad (m > 0) \tag{3.20}$$

and its angular velocity

$$\mathscr{A}_{\alpha\gamma p}\,\omega^A_\gamma a^3 = B_{\beta 0}[BA]^{\alpha p}_{\beta 0}-A_{\beta 0}(AB)^{\gamma m}_{\beta 0}[BA]^{\alpha p}_{\gamma m}+\dots \quad (m > 0). \tag{3.21}$$

The third term in equation (3.20) could be easily evaluated exactly, so that this was done, but only a few terms of the next term were obtained. The final result is written as

$$U^A_\alpha = V^A_\beta(K\delta_{\alpha\beta}+Ln_\alpha n_\beta)+bV^B_\beta(M\delta_{\alpha\beta}+Nn_\alpha n_\beta), \tag{3.22}$$

where

$$\left.\begin{aligned}
K &= 1+17ab^5/16+\sum_{m>0}(ab^{2m+1}/48)\{54b^4-6a^2b^2(4m^2+6m-1)\\
&\qquad +a^4(m+1)^2(2m+1)(2m+3)\},\\
L &= (15ab^3/4)+15ab^5/16-15a^3b^5/2+\sum_{m>0}(ab^{2m+1}/48)\\
&\qquad \times\{54b^4-18a^2b^2(4m^2+18m+9)\\
&\qquad +a^4(m+1)(m+3)(2m+1)(2m+3)\},\\
M &= \tfrac{1}{4}(3+a^2+b^2)+a^3b^3\{f_{11}+(a^2+b^2)f_{12}\},\\
N &= \tfrac{3}{4}(1-a^2-b^2)+a^3b^3\{g_{11}+(a^2+b^2)g_{12}\}.\\
8f_{11} &= -(3b^2+5a^2)(3a^2+5b^2),\\
8(g_{11}+f_{11}) &= 150-420(a^2+b^2)+6(63a^4+63b^4+130a^2b^2),\\
4f_{12} &= (5a^2+3b^2)\{5-\tfrac{1}{4}(39a^2+77b^2)+\tfrac{15}{28}(5a^2+7b^2)(7b^2+3a^2)\},\\
4(g_{12}+f_{12}) &= 195-3(185a^2+249b^2)+3(173\tfrac{4}{7}a^4+482a^2b^2+301b^4)\\
&\qquad -(15/7)(5a^2+3b^2)(3a^2+7b^2)(7b^2+5a^2).
\end{aligned}\right\} \tag{3.23}$$

* This is a particular solution. Other solutions lead to the same results

† The solution for the forces in terms of the velocities is not only less useful but much more slowly convergent.

The distance between centres is chosen as the unit length so that $R = 1$ and a, b now measure the radii as fractions of R.

Because some of the coefficients of the higher powers of a and b are rather large, it seemed worth while to obtain an approximate solution containing contributions from all the terms in equation (3.20). Such a solution can be obtained by assuming that the constants $A^{\alpha n} = 0$, for all $n > 1$; the various sums simplify and can be evaluated.

The answer obtained in this way yields the values

$$\left.\begin{aligned} K &= 1-(3ab^3/4)\,Q(b^2+\tfrac{5}{3}a^2), \quad (1/Q) = 1-\tfrac{1}{4}a^3b^3(5-8a^2-8b^2)^2,\\ L &= -(15ab^3/4)\,P(1-a^2-\tfrac{3}{5}b^2), \quad (1/p) = 1-a^3b^3(5-6a^2-6b^2)^2,\\ M &= \tfrac{1}{4}(3+a^2+b^2)+(3a^3b^3/8)\,Q(b^2+\tfrac{3}{5}a^2)\,(b^2+\tfrac{5}{3}a^2)\,(5-8a^2-8b^2),\\ N &= \tfrac{3}{4}(1-a^2-b^2)+(3a^3b^3/8)\,[10P(1-a^2-\tfrac{3}{5}b^2)\,(1-b^2-\tfrac{3}{5}a^2)\,(5-6a^2-6b^2)\\ &\qquad -Q(b^2+\tfrac{3}{5}a^2)\,(b^2+\tfrac{5}{3}a^2)\,(5-8a^2-8b^2)]. \end{aligned}\right\} \tag{3.24}$$

The approximation $P = Q = 1$, yields terms which have already been included in the earlier expressions (3.22) and (3.23) so that the corrections due to higher order terms are proportional to $(P-1)$ and $(Q-1)$. These corrections are small unless one sphere is very much larger than the other. It seems likely, therefore, although it is not certain, that higher-order terms in the expansion (3.20) are not appreciable unless one sphere is much larger than the other, and that the series expansion is useful even up to the point where the two spheres are almost in contact.

4. Discussion for two particles

The analysis of the fall of two particles can be discussed without reference to the work of the previous two sections and the results of the discussion compared with the available experimental results. With the aid of the calculations of those two sections we can evaluate the constants and compare directly these results with our theory.

The theoretical calculations of Smoluchowski and Burgers should agree satisfactorily with experimental results on the fall of two equal spheres when the distance between centres is more than three times their diameter. For smaller distances between centres the only theory available is that of Stimson & Jeffery, for spheres falling behind one another with equal velocities. In this section we consider some general implications of our calculations and extend the comparison with experiment not only for equal spheres but for unequal spheres, as far as that is possible at present.

From the linearity of the hydrodynamic equations we conclude that the particle velocities can be derived from the vector equations,

$$\left.\begin{aligned} U^A_\alpha &= V^A_\beta(K\delta_{\alpha\beta}+Ln_\alpha n_\beta)+b\,V^B_\beta(M\delta_{\alpha\beta}+Nn_\alpha n_\beta),\\ U^B_\alpha &= V^B_\beta(K'\delta_{\alpha\beta}+L'n_\alpha n_\beta)+a\,V^A_\beta(M'\delta_{\alpha\beta}+N'n_\alpha n_\beta), \end{aligned}\right\} \tag{4.1}$$

where the velocity coefficients K, K', ..., N' denote functions only of the radii and the distance between centres, approximate values of which are given in the previous section. In the limit of infinite separation K and K' tend to unity and the other constants tend to zero.

If the velocities are resolved along and perpendicular to the line of centres, these equations become

$$\left.\begin{aligned} U_p^A &= (K+L)\,V_p^A + b(M+N)\,V_p^B, \\ U_n^A &= KV_n^A + bMV_n^B, \end{aligned}\right\} \tag{4.2}$$

together with two similar equations for the velocity of B. Thus the velocity coefficients can be obtained by solving first the problem where both particles are acted on by forces along the line of centres, and then by solving a second problem where the forces are perpendicular to the line of centres. The first of these two problems has been solved in the most important case of equal velocities by Stimson & Jeffery, but a solution of the second has not yet been obtained. The values given below for these coefficients are determined from our series expansion.

Experiments are usually conducted with particles falling under gravity in a vertical plane (figure 1). If the line of centres is inclined at an angle θ to the horizontal, the horizontal and vertical velocities are

$$\left.\begin{aligned} U_z^A &= V^A(K+L\sin^2\theta) + bV^B(M+N\sin^2\theta), \\ U_x^A &= (V^AL + bV^BN)\sin\theta\cos\theta, \\ U_z^B &= B^B(K'+L'\sin^2\theta) + aV^A(M'+N'\sin^2\theta), \\ U_x^B &= (aV^AN' + L'V^B)\sin\theta\cos\theta. \end{aligned}\right\} \tag{4.3}$$

Equal particles with the same Stokes velocities V have the same velocity components

$$\left.\begin{aligned} (U_z/V) &= (K+aM) + (L+aN)\sin^2\theta, \\ (U_x/V) &= (L+aN)\sin\theta\cos\theta. \end{aligned}\right\} \tag{4.4}$$

Hall (1956), having taken great care to manufacture almost identical particles, has verified that the velocity and the inclination θ of the line of centres are constant during the motion of equal particles. The above equations suggest further conditions on the velocity which are independent of the values of the constants. First, the vertical velocity, for a given distance between centres, should be a linear function of $\sin^2\theta$. Secondly, the particles move away from the vertical with a velocity proportional to $(L+aN)$ which is the slope of the graph of the horizontal velocity. The greatest transverse velocity is $\frac{1}{2}(L+aN)$ when $\theta = 45°$, or one half of the difference between the vertical velocities of fall for $\theta = 0$ and $\theta = 90°$.

The only experimental data on pairs of spheres available at the moment is that of Hall (1956) and, owing to the large variation of the viscosity of the liquid with temperatures and the possibility that circulation of the liquid was not entirely prevented, his values may not be accurate enough to judge the theory.

Figure 2 shows, for one pair of spheres,* the speed of fall plotted against $\sin^2\theta$ for various values of the ratio $2a = d/R$ of particle diameter to the distance

* These are the particles labelled by Hall P 3. The other experimental data given here relate to the same pair. Curves drawn for a second pair of particles are very similar and differ from the theory in the same way.

between centres. This diagram does not confirm a linear relation but the experimental errors are such that a relationship of this kind is not denied.

Assuming that it exists, the sum $m = (L+aN)$ equals the slope of the line, for each value of (d/R). Slopes calculated by two methods are given in table 1. The first is calculated by a least squares method, taking

$$10m_1 = \{8(U_{90}-U_0)+4(U_{60}-U_{30})\}/V_\infty.$$

The second is calculated by the simple formula

$$m_2 = (U_{90}-U_0)/V_\infty.$$

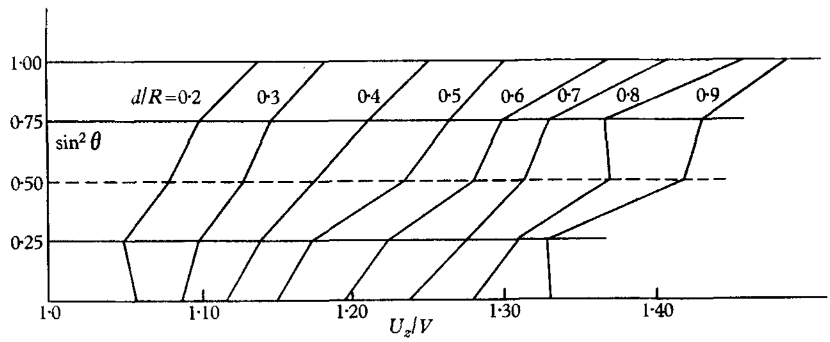

FIGURE 2. Speed of fall of equal spheres for various ratios of diameter to distance between centres (experimental).

Finally, the table includes the values of a third quantity determined by the experimental data. The measurements include the angle ϵ between the direction of fall and the vertical. According to equations (4.4), when ϵ is small

$$\tan\epsilon = \frac{U_x}{U_z} = \frac{(L+aN)\sin\theta\cos\theta}{(K+aM)+(L+aN)\sin^2\theta}$$

$$= (L+aN)\sin\theta\cos\theta/(K+aM), \quad \text{approx.}$$

Now $(K+aM) = (Uz)_0/V$, where $(U_z)_0$ is the vertical velocity when $\theta = 0$. Thus, applying the equation for $\tan\epsilon$ when $\theta = 45°$, we deduce that

$$m_3 = 2\frac{(U_z)_0}{V_\infty}\tan\epsilon = (L+aN).$$

These three quantities agree as well as can be expected and, in particular, they all vary in the same way with the distance between centres.

Figure 3 shows the same data in a different form, the velocity of fall being plotted against (d/R) for various directions of the line of centres for the pair of particles. These curves have kinks which may be due to experimental error (although it is curious that they appear in the same places for all θ) and it is noteworthy that they are continuous up to the value of $d/R = 1$, which corresponds to contact between the spheres.

Only two theoretical curves, for $\theta = 0$ and $\theta = 90$, are drawn for comparison, the others being very similar, and it is clear that agreement with experiment is not very good. This may well be due to a deficiency in the theory, but it may again be due to experimental error. Thus, in the case of large separation, where the theory is unlikely to be in error, the experimental values are some 20 % less than the theoretical values.

d/R	m_1	m_2 (experimental)	m_3	$m = L+aN$ (theoretical)
0·2	0·065	0·056	0·071	0·073
0·3	0·10	0·098	0·098	0·105
0·4	0·14	0·136	0·131	0·133
0·5	0·16	0·153	0·150	0·152
0·6	0·17	0·173	0·161	0·163
0·7	0·16	0·170	0·161	0·165
0·8	0·16	0·166	0·154	0·161
0·9	0·17	0·158	0·149	0·154
1·0	0·16	0·152	0·138	0·155

TABLE 1. Comparison of theoretical and experimental values of the slopes of the lines shown in figure 2, for various values of (d/R).

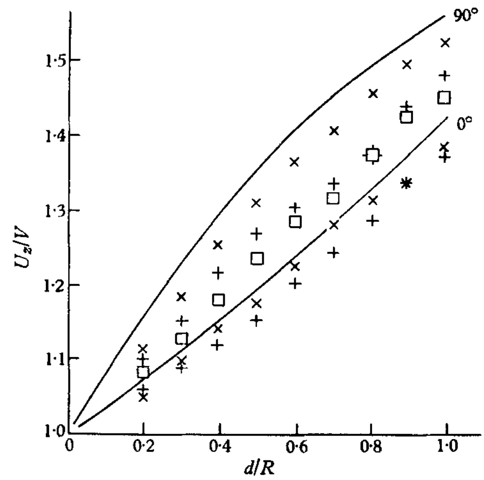

FIGURE 3. Velocity of fall of equal spheres plotted against the distances between centres, for various inclinations θ of the line of centres to the horizontal. [Theoretical – full lines for $\theta = 0$ and 90°: Experimental + (0°), × (30°), □ (45°), + (60°), × (90°)].

The theoretical curves were derived from the expressions given in the previous section written in the form,

$$1024K = 1024 - 17d^6 - 5d^8 - (69/16)\,d^{10} - (31/8)\,d^{12},$$

$$1024L = -240d^4 + 105d^6 + 47d^8 + (233/16)\,d^{10} - (33/16)\,d^{12},$$

$$1024aM = 384d + 64d^3 + 6d^{11} - (113/10)\,d^{13} + (225/28)\,d^{15},$$

$$1024aN = 384d - 192d^3 + 150d^7 - 15d^9 - (471/2)\,d^{11} + (176\tfrac{9}{35})\,d^{13} - (40\tfrac{5}{28})\,d^{15},$$

which yield the values of the velocity constants as a function of d ($R = 1$) given in table 2.

d	K	L	aM	aN
0·0	1·0	0·0	0·0	0·0
0·1	1·000	−0·000	0·038	0·037
0·2	1·000	−0·000	0·076	0·074
0·3	1·000	−0·002	0·114	0·107
0·4	1·000	−0·006	0·154	0·138
0·5	1·000	−0·013	0·195	0·165
0·6	0·999	−0·025	0·239	0·188
0·7	0·998	−0·041	0·284	0·207
0·8	0·994	−0·060	0·332	0·221
0·9	0·986	−0·075	0·384	0·229
1·0	0·965	−0·073	0·440	0·228

TABLE 2. Theoretical values of K, L, aM and aN of equations for values of (d/R) between 0 and 1.

5. Three-particle interactions

In this section we consider the interactions between particles when three or more are falling through a liquid. In addition to the interactions between pairs of particles, previously considered, there are now some that involve three distinct particles, namely, the effect of one particle on another in the presence of a third, and even more complicated interactions. Although these interactions arise in higher order approximation than the two-body interactions, it will appear that they are not always negligible. They can be very numerous and they may be important for certain configurations of the particles. Thus the main correction to the fluid motion around B due to a particle C distant S from it is proportional to (V^C/S^2) (cf. equation 3.10). As this is a dipole correction, it gives to a third particle A a velocity of order V^C/R^2S^2, where $R = AB$. In some circumstances, therefore, this term is quite important. In this section it is not our intention to evaluate these interactions to high order, or to discuss the general sedimentation problem, but rather to examine a few of the interactions in detail. Until methods of calculating the particle motions are discovered, which are more satisfactory than those used so far, there is little point in calculating these complicated terms to very high accuracy.

The general formulae are easily obtained by extending equations (3.15) to (3.20) so that they apply to more than two particles. The general formula for the velocity of any particle A is

$$aU^A_\alpha = aV^A_\alpha + bV^B_\beta(BA)^{\alpha 0}_{\beta 0} - \sum_{C \neq B} cV^C_\beta \sum_{B \neq A} (CB)^{\gamma m}_{\beta 0}(BA)^{\alpha 0}_{\gamma m} + \dots \quad (m > 0). \quad (5.1)$$

In this equation the sum over B includes all particles other than A; and the sum over C includes all particles other than B, including $C = A$, so that this expression reduces to that already given in equation (3.24) when there are only two particles. The second term is the vector sum over all particles of the main term in their two-body intersection with A. This is the term which is normally considered in an aggregate of particles. On account of its long range it is responsible for the shielding effect (Kynch 1954) and tends to make all the particles move together, but the details of the effects of this term have still to be worked out. Except for special arrangements it is difficult to deduce the motion of just three particles, even when this two-body interaction is the only one included. One special arrangement is that where one particle A is in a vertical plane midway between the others. If the three particles have the same Stokes velocity, the two particles separate to allow A to pass between them and then close up behind it. This motion is stable with respect to small displacements of A on either side of the vertical plane.

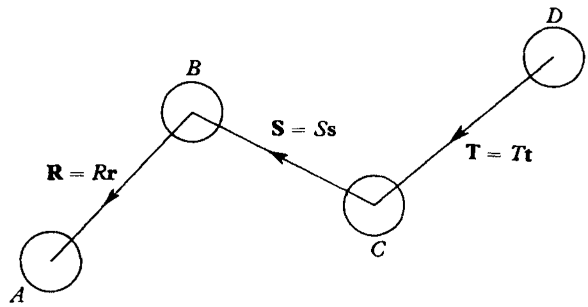

FIGURE 4. Co-ordinates for four falling spheres.

The third term contains three-body interactions, those which involve three distinct particles (CBA) and that due to the reaction of (ABA) of A on itself due to the presence of B. Interactions involving four distinct particles appear first in the next term of the series. The main term in the velocity of A due to C in the presence of B, using a notation shown in figure 4, is

$$(CBA) \quad U^A_\alpha = (c/a)\, V^C_\beta (CB)^{\gamma p}_{\beta 0}\, (BA)^{\alpha 0}_{\gamma p} \quad (p = 1). \tag{5.2}$$

Since this expression is symmetrical in α and p, and the principal terms of $\frac{1}{2}(BA)^{\alpha 0}_{\gamma p} + \frac{1}{2}(BA)^{\alpha 0}_{p \gamma}$ are proportional to BA_α, it follows that the principal terms lead to a correction directed along the line BA. They give

$$(CBA) \quad U^A_\alpha = (15cb^3/8R^2S^2)\,(3\cos^2\theta - 1)\,(V^C_\beta s_\beta)\, r_\alpha. \tag{5.3}$$

Its dependence on the relative positions of the particles is not too complicated. Let us assume the $\mathbf{V}^c$ is directed vertically downwards. The correction is directed along the line AB and, when C is below B, it is from A towards B as long as $3\cos^2\theta > 1$, i.e. the line AB lies within a cone with BC as axis and semi-vertical angle $\alpha = \cos^{-1}(\frac{1}{3})$. If C is above B the sense is reversed. Hence, for certain orientations the velocity (CBA) increases the main term (BA), whereas for other orientations it tends to cancel it. The change of sign when the direction of CB is reversed means that the three body terms tend to cancel under certain circumstances. For example, when a set of particles falls downwards in a line, these terms tend to cancel for particles in the middle, to increase the speed of those at the top and to decrease the speed of those at the bottom, i.e. to keep the particles together. (They are, of course, usually rather less in magnitude than the two-body interactions which tend to separate the particles.)

The reaction of A on itself due to the presence of B is obtained by making $C = A$ and $\theta = 180°$. It is, as given in equation (3.14),

$$(ABA) \quad U^A_\alpha = -(15ab^3/4R^4)\,(V^A_\beta r_\beta)\, r_\alpha. \tag{5.4}$$

The correction of next importance comes from the remaining parts of (5.2) and the next term in (5.1) with $m = 2$

$$\left.\begin{aligned}(CBA)\quad \mathbf{U}^A = {} & \{3cb^3(3b+5a^2)\}\,(\mathbf{V}.\mathbf{s})\,\{(1-5\cos^2\theta)\,\mathbf{r}+2\cos\theta\mathbf{s}\}\\ & +\{3cb^3(3b^2+5c^2)\}\,\{(\mathbf{V}.\mathbf{s})\,(1-5\cos^2\theta)\,\mathbf{r}+2(\mathbf{V}.\mathbf{r})\cos\theta\mathbf{r}\}\\ & -(cb^5/64R^3S^3)\,[\mathbf{V}(117\cos^2\theta-49)-(\mathbf{V}.\mathbf{r})\,54\cos\theta\mathbf{s}\\ & +(\mathbf{V}.\mathbf{s})\,(1575\cos^3\theta-729\cos\theta)\,\mathbf{r}\\ & -\{(\mathbf{V}.\mathbf{s})\,\mathbf{s}+(\mathbf{V}.\mathbf{r})\,\mathbf{r}\}\,(315\cos^2\theta-93)].\end{aligned}\right\}\quad (5.5)$$

This expression reduces, when $C = A$, to the expression given in equation (3.27)

$$(ABA)\quad \mathbf{U}^A = (15a^3b^3/2R^6)\,(\mathbf{V}.\mathbf{r})\,\mathbf{r}-(ab^5/16R^6)\,\{17\mathbf{V}+15(\mathbf{V}.\mathbf{r})\,\mathbf{r}\}. \quad (5.6)$$

If the position of C is changed so that the direction of $\mathbf{s}$ (or $\overrightarrow{CB}$) is reversed, the terms in the first two lines of (5.5) change sign but those after do not. Let us compare with (5.6) the correction when C is diametrically opposite to A, though the same distance from B as A, so that, when $a = c$,

$$\mathbf{U}^A = -(15a^3b^3/2R^6)\,(\mathbf{V}.\mathbf{r})\,\mathbf{r}-(ab^5/16R^6)\,\{17\mathbf{V}+159(\mathbf{V}.\mathbf{r})\,\mathbf{r}\}. \quad (5.7)$$

Comparing (5.6) and (5.7) we see that the coefficients of the third line are sufficiently large to produce an appreciable and complicated variation with the configuration of the particles. This is not unexpected since, as shown in the previous paragraph, there is a considerable variation in the two-body correction with orientation which must be reflected in these more complicated corrections.

Without comment, we give the first four-body correction, which is directed along the line BA

$$(DCBA)\quad dV^D_\beta (DC)^{\delta q}_{\beta 0}\,(CB)^{\gamma p}_{\delta q}\,(BA)^{\alpha 0}_{\gamma p} = \frac{75ab^3c^3d}{16R^2S^3T^2}\,(\mathbf{V}^D.\mathbf{t})\,r_\alpha$$
$$\times[1-3(\mathbf{r}.\mathbf{s})^2+3(\mathbf{t}.\mathbf{s})^2\{5(\mathbf{r}.\mathbf{s})^2-1\}-6(\mathbf{r}.\mathbf{s})\,(\mathbf{s}.\mathbf{t})\,(\mathbf{t}.\mathbf{r})]. \quad (5.8)$$

6. Comments on sedimentation

This section contains some general remarks on the calculation of sedimentation velocities with small concentrations of particles, which follow naturally from the results of the previous sections. In § 5 a few comments were made about the role of the two-body interaction in an assembly of particles; here we are concerned with the higher order corrections.

A finite assembly of identical particles falling freely under gravity through a viscous medium cannot do so in the form of a regular lattice array. Differences between the motion of particles near the edge and those near the centre of the assembly produce perturbations which spread inwards from the boundary and spoil the regularity. On the other hand, the shielding effect mentioned earlier tends to prevent differences in velocity between particles, and it could happen that there is short-range order amongst the particles even when long-range order is absent. If there is short-range order this must be allowed for in any calculation of high-order interactions. If it is absent the mean corrections can be calculated by taking an average for a random distribution of particles.*

With this in mind, let us now examine the various terms of equation (5.1), assuming first that particles near a given particle A form a regular array over

* We neglect the effect of aggregation and other complicating factors, such as wall effects.

distances for which the corrections are appreciable. It can be shown that many terms vanish independently of the form of the array. To prove this we note that all the terms start in the same way,

$$\sum_{C \neq B} c V_\beta^C (CB)_{\beta 0}^{\gamma m} \ldots$$

Now $(CB)_{\beta 0}^{\gamma m} = (-)^m (BC)_{\beta 0}^{\gamma m}$, which changes sign when m is odd (though not when m is even) if the position of C is changed so that BC is reversed in direction. But, in a lattice array, all the particle C can be taken in pairs on opposite sides of B; hence all those terms are zero where m is odd. For lattices the first relevant value of m is 2. The corrections given in the previous section in equation (5.4) can be ignored and only the third line of equation (5.5), which comes from $m = 2$, need be considered, and this is of order d^6, where d is the ratio of particle diameter to distance between centres. Summing over all particles leads to a term in the velocity or order (concentration)^2.

This argument does not apply when there is no order. To prove this we consider the three-body terms only, although the same method can be applied to the other terms, where it takes a much more complicated form. The three-body interactions are either of the type (CBA) or (ABA),

$$\sum_{C \neq A, B} c V_\beta^C (CB)_{\beta 0}^{\gamma m} (BA)_{\gamma m}^{\alpha 0} + \sum_{B \neq A} a V_\beta^A (AB)_{\beta 0}^{\gamma m} (BA)_{\gamma m}^{\alpha 0}.$$

In the absence of correlations, for given positions of A and B, we now average over all positions of C. Neglecting the finite size of the particle, the first term is zero for all $m > 0$, owing to the angular variation of $(CB)_{\beta 0}^{\gamma m}$. Owing to the finite size of the particles, however, C is excluded from a region near the particle A of radius $2a$. The average value of $(CB)_{\beta 0}^{\gamma m}$ when C is in this region is approximately $(AB)_{\beta 0}^{\gamma m}$ and the probability of its being there is $8c$, where c is the volume concentration of particles. Thus the correction becomes approximately

$$(1 - 8c) \sum_m a V_\beta^A (AB)_{\beta 0}^{\gamma m} (BA)_{\gamma m}^{\alpha 0}.$$

The first term of this sum with $m = p = 1$, namely $aV_\beta (AB)_{\beta 0}^{\gamma p} (BA)_{\gamma p}^{\alpha 0}$, already given in equation (3.12), was, in fact, used by Burgers in his calculation of the sedimentation velocity (1942). It has been criticized (Hawksley 1950) on the grounds that it gives a correction which is too small. The arguments given here suggest precisely the opposite: that it overestimates the correction when there is a random distribution of particles and is much too large when there is partial order. Assuming that the basis of these calculations is correct we must attribute the error to neglect of higher order correctives which have large coefficients.

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